Luck, logic, and white lies: The mathematics of games, second edition, by Jörg Bewersdorff

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As the title indicates, Bewersdorff's book is intended to span the mathematics of *games* in general – not only games of chance but also including strategic and skill games. The author covers all the big categories of games – casino, tournament, and house or social games. In fact, the skill-strategic dimension of the games balanced with the chance-uncertainty dimension is the central element around which the author presents games as an important field of application of mathematics; he takes them as a good opportunity to advocate for the beauty and power of mathematics. To that point, the book is written so as to be both popular and scholarly, and these attributes are not at all inconsistent with each other for such a general topic, content, and style.

The popular attribute is mainly provided by the problem-oriented selection of topics and sub-topics and the organization of the content: Each chapter starts with a nice relevant problem, general question, or curiosity that mathematics is called upon to solve and can be read independently, like a collection of articles in a mathematics magazine. The presentation is very often in the style of Martin Gardner's recreational mathematics; in fact, some of the games analysed in the current book (Tic Tac Toe, Hex, Go) are also present in Martin Gardner's influential book *The Scientific American Book of Mathematical Puzzles and Diversions* (1959), of course using more advanced mathematical concepts than the latter. The historical approach, present in almost all topics, whether we talk about the games themselves or the mathematical concepts or theories dealing with them, also counts toward the popular attribute.

The scholar attribute is given firstly by the general principle driving the problem-solving approach throughout the book, namely to show how in this field of application the three main branches of mathematics – game theory, probability theory, and combinatorics – collaborate toward describing the games, strategies, and optimal play. Further, some concepts and theoretical results (especially from game theory) are relatively advanced and usually not employed in most of the published books on mathematics of gambling. Adding to these elements is that the game sections benefit by relevant citations and are packed with further-reading sections and chapter notes. What we have here, then, is a kind of textbook on a popular subject addressed in a popular style.

In the preface, the author provides a basic taxonomy of games with respect to the three elements he identifies as causing the uncertainty and driving the mechanism of every game: chance, the large number of combinations of different moves, and different levels of information among the individual players. These elements yield three basic categories of games, namely combinatorial games, games of chance, and games of skill – which are not mutually exclusive – there are games that do fall in two categories or even in all three (like poker and Skat). Still in the preface, the author describes the relationship between mathematics and games in terms of both mathematical applications and the interests and behavior of the players, and emphasizes the roles of game theory, probability theory, and combinatorics. It is remarkable that in this section the author talks about the central notion of probability as being a *measure* (of the certainty with which a random event occurs) in the common sense, while explaining from the beginning that probability is not

all that counts toward strategy and optimal play. He further introduces a definition for the Laplacian probability in the first game section (Dice and Probability).

Unlike the customary structure of a gaming mathematics book, this book does not have a systematically organized supporting-mathematics chapter preceding the descriptions and analyses of the games. In fact, none of the mathematics presented is systematized as for a student's or player's study or use; it is simply applied for the gaming-related problems and questions posed across the game sections. The mathematical level differs among the different chapters, but not increasingly. A new concept is introduced when needed and references back to previous concepts and explanations are present across the chapters. This does not mean that chapters cannot be read quite independently, but definitely not by a non-mathematical reader.

The Laplacian probability and the associated basic notions are introduced in the first three chapters (Dice and Probability, Waiting for a Double 6, and Tips on Playing the Lottery: More Equal than the Equal?), by using combinatorial examples from dice, lottery, and poker games. These examples are also used to explain the rules of combinatorial calculations (including the use of the Pascal triangle) and to show the average reader how the multiplication power of combinations affects the probability of the various events in combinatorial games of chance. This is an important lesson for gamblers in regard to the erroneous or subjective quick estimations of probabilities of winning (usually overestimated). Of course, the author chose the best game for this didactical purpose, namely lottery, and made a comparison between the lottery combinations of numbers and card combinations in poker. Conditional probability is introduced in Chapter 9.

In the historical side of this introductory part, the author does not miss the opportunity to tell the reader the well-known story about the birth of probability theory, originating in a problem from a game of chance (De Méré problem), debated in a correspondence between Pascal and Fermat at the middle of the 17th century.

The law of large numbers (LLN) is introduced in Chapter 5 and explained through experimental examples, with reference to its history (a first version of LLN was formulated and proved in 1690, before probability theory was crystallized!) and to the strong law of large numbers. What is remarkable is that the author, still in an experimental-example framework, brings into discussion the sensible issue of the stabilization of the relative frequencies, which is a core question in philosophy of probability, linking the empirical-experimental and the abstract-theoretical (*What* makes the relative frequency become stabilized around mathematical probability?)[1] This is not the only philosophical-foundational aspect touched on in this book in regard to probability theory. The issue of a mathematical objective description of randomness and of the equally-possible attribute in the Laplacian definition of probability, the conciliation between chance (as the object of investigation of probability theory) and mathematical certainty are discussed in the same historical style, in the context of physics. This contextual choice (in Chapter 8) in a book explaining the mathematics of games is not at all something beyond the topic. From kinetic gas theory in the 19th century to quantum mechanics in the 20th century, physics has shown that the non-deterministic approach makes randomness *objectively* present in our theories, since all such theories were tested

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[1] In other non-Kolmorogovian formal accounts of probability, the stabilization of the relative frequency is an axiom/postulate, conventionally taken to be true.

true in physics, both theoretically and experimentally. As such, randomness is not just an expression of our ignorance or lack of knowledge, as reflected in the Laplacian definition of probability, but a theoretical necessity. Indeed, since the entirety of the state parameters determining a theoretical model of a system can never be precisely measured, predictions about the future behaviour of that system are always subject to randomness. Still, the laws of averages that the probabilistic approach provides turn into a mathematical description of an "average behaviour" of the system that can be accommodated with the deterministic approach. On the other side, mathematics has shown that a model for probability can be constructed without a mathematical definition of randomness, from Bohlmann's to Kolmogorov's axiomatic system on which standard probability theory relies. But, one may ask, what do games have to do with kinetic gas theory and quantum mechanics? The author explains this relationship by introducing the theoretical concept of a *chaotic* system as an intermediary concept between the physics and mathematics of random processes and phenomena: By placing the games in the realm of classical mechanics where unpredictable outcomes are physical, games can be characterized by the fact that causal relations are highly sensitive to small perturbations that cannot be measured ("small causes, large effects") – in other words, they are chaotic systems – and as such, "physical" randomness becomes a sort of mathematical randomness. Put this way, the certainty of mathematics and the uncertainty of randomness and probability no longer appear to exclude each other radically. It is an original preamble for the introduction of the Kolmogorovian axiomatic system defining probability (also in Chapter 8) Although the chapter ends with an elementary description of this axiomatic system, I would say that this is "the most" scholarly chapter in this book, seemingly addressed to graduates in philosophy and history of science. However, this topic is an ideal completion for the association of the terms *luck* and *logic* (even from the book's title) in the strategic discussions in regard to games, one of the main focuses of this work.

Sometimes classical problems from recreational mathematics are discussed, apparently not related to the described games (e.g., Buffon's Needle problem or the three-door problem). The goal seems to be that of making the reader aware of how tricky or counterintuitive probability theory may be sometimes, with respect to both its application and interpretation, and how the same concepts (both objective and subjective) of randomness and equiprobable are responsible for that.

The author does not limit the presentation of probability theory to elementary notions, foundation, and discrete probability, but extends it to advanced concepts when a description of a game requires it. Markov chains are introduced in the discussion of the game of Monopoly. The transition from one square to another in Monopoly provides a good elementary example to start with when teaching someone about a Markov chain as a sequence of random trials with finite sets of outcomes in which the probability that an event occurs on the $(n + 1)$-th trial depends only on the event of the n-th trial. The Monopoly example also allows an easy generalization to other less popular games, namely Snakes and Ladders, and is applied as well to the classical Ruin Problem.

At the end of Part I (Games of Chance), Blackjack has the lion's share. It is to the author's credit that he has comprised a typical 200-page book on blackjack mathematics and strategy into a 20-page chapter with essential information focused on optimal play. By using only the basic notions of probability, conditional probability, and expectation,

the author explains the rules of the optimal play in blackjack and the mathematics behind the High-Low count of cards. The numerical results of the most frequent gaming situations (probabilities, expectations, and criteria for optimal play) are grouped efficiently in general tables in a much smaller number than in other books dedicated to blackjack.

We can say fairly that this is a chapter that exclusively addresses the gambler – the mathematics is elementary and easy to follow, and the aspects of optimal play and strategy are discussed in the player's favour.

Still in the part dedicated to games of chance, the author presents the Monte Carlo Method of simulating plays where chance is involved, as a long series of trials, as a convenient alternative to the mathematical deduction of excessively complex formulas for probabilities and statistical indicators. It is a good opportunity for making the connection with programming, which is more involved in the parts dedicated to combinatorial and strategic games, some of them being discussed through an algorithmic approach. The advantages of the Monte Carlo Method in analysis of results and decision-making are clearly specified, and the brief history of this method is presented, as we have come to expect by now. In the same chapter, the author talks about the generation of random numbers – necessary for the Monte Carlo simulations and for the contemporary electronic games of chance – and reveals the mathematical reasons for the accuracy and efficiency of such "algorithmic randomness" which relates to the primes and their properties. Still related to the Monte Carlo Method, the notion of sample function is introduced at the end of this chapter with the simple example of die throwing.

In Part II (Combinatorial Games), chess is given (unexpectedly) only ten pages, but it is discussed only as related to Zermelo's theorem (applicable to chess and other comparable games), which stands as the main principle of chess programming. It is the most general (and shortest) mathematical description of chess, and the author recalls this theorem (also called the *minimax theorem*) in Chapter 27, dedicated to chess programming.

By contrast, Go is given the largest space and is discussed in terms of advanced concepts of combinatorial game theory from the works of Conway, Milnor, and Hanner. It is an excellent example of how a general mathematical theory was developed by analysing old classical positional games like Go.

In Part III (Strategic Games) a focus is given to poker as a game with imperfect information, characterized by both chance and skill, a perfect example to analyse mathematically whether the mixed strategies are possible. While not getting into the details of poker and its explicit strategies, the author answers fundamental questions regarding optimal play by citing the works of von Neumann, Morgenstern, and Farkas. The conclusion is that optimal play in poker is possible, according to the available mathematical models, only under the assumption (idealization) that strategies chosen by players are random and the probabilities with which the various strategies are chosen determine the degree of success that a player can expect on average. Within this theoretical framework, the psychology of poker (including bluffing) is reflected only as a probabilistic element and not as a deterministic one. The elaboration and calculation of minimax strategies for poker require universal algorithms that are introduced in the next two chapters 36 and 37 – linear optimization, the simplex algorithm, and the rectangle rule.

The analysis of poker is resumed in Part IV, which is elaborated as an epilogue to the discussion on the relationships between chance, skill, and symmetry with respect to strategy and optimal play. The author's general conclusion, supported by the results of game theory, is that only the combinatorial elements of a game bring about a causality that can be influenced by skill, and the chance-logic-bluff classification of the game elements does not count in giving skill roles that can be reflected or quantified in the mathematical models we use under idealized conditions for deriving optimal play. As such, pure games of skill are without exception combinatorial games. Still, the author further addresses fundamental questions regarding skill such as "How can a system for the influence of skill in a game be defined?" (the topic of Chapter 49); "What suggestions have been made hitherto for measuring the influence of skill on a game?" (the topic of Chapter 50); and the hotly debated popular question regarding poker "Is poker a game of chance?"[2], which remains open and concludes the book.

Epiloguing in this way, the book leaves the impression of its author's being a skilled advocate of the unlimited power of mathematics, shown through the examples of games. Not only is mathematics able to describe the games and the way we play them, but it is entitled to address fundamental questions beyond the problem-solving aspects of games and gaming. It is mainly game theory and probability theory that grant mathematics such a virtue. In particular, game theory provides the kind of *generality* that allows the mathematical treatment of *all* categories of games with respect to strategy and optimal play, and it is the author's merit for achieving such coverage in only 561 pages, including the math and math history lessons. Although the chapters can mostly be read independent of each other, and the mathematical content is not systematized throughout the book, the mathematically-inclined reader can put things together to have an objective overview of one of the most interesting fields in application of mathematics – games – which themselves shaped the development of mathematics. As for the average player, even the title itself suggests that the ingredients of a game and any strategy of playing it are of radically different natures, and mathematics can deal with them together. Passing from the abstract mathematics of games to the real play in the form of application is not so much a matter of mathematical skill as of interpreting empirically the mathematical concepts and theoretical results. In the realm of games, such empirical interpretation is many times at least subjective, and it is not the object of investigation of pure mathematics.

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[2] Poker is lying in author's taxonomy of games in the preface in a common zone for combinatorial games, strategic games, and games of chance; however, as per the arguments in the last chapter, this placement may change with the variations of poker and their rules.